SOME PROPERTIES OF POLYNOMIAL SETS OF TYPE ZERO

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1. **Introduction.** Pincherle,[1] in his study of the difference equation

$$\sum_{n=1}^{k} c_n \phi(x + h_n) = f(x),$$

was led to consider a set of Appell polynomials, in infinite series of which solutions could be represented. We considered the same equation[2] by means of a different Appell set, the change resulting in a significant alteration of the regions of convergence (for the series). This permitted an enlargement of the class of functions $f(x)$ for which a solution could be shown to exist. Recently we treated the more general equation[3] (linear differential equation of infinite order)

$$L[y] \equiv a_0 y + a_1 y' + \cdots = f(x),$$

where, under suitable conditions on L and f, a solution was found. Here, too, it was possible to relate the equation to a corresponding problem of expanding functions in series of Appell polynomials. It is this close relation to functional equations that adds interest to the study of Appell sets.

As is well known, Appell sets $\{P_n(x)\}$ $(n = 0, 1, \cdots)$ are characterized by either of the equivalent conditions

$$P_n'(x) = P_{n-1}(x) \qquad (P_n \text{ a polynomial of degree } n); \tag{1.1}$$

$$A(t)e^{tx} \cong \sum_{0}^{\infty} P_n(x)t^n, \tag{1.2}$$

where $A(t) \cong \sum a_n t^n$ is a formal power series, and where the product on the left of (1.2) is formally expanded in a power series in accordance with the Cauchy rule. We shall say that the series $A(t)$ is the *determining series* for the set $\{P_n\}$.

For the particular equation

$$y(x + 1) - y(x) = f(x),$$

Pincherle used the Appell set with $A(t) = 1/(e^t - 1)$, getting essentially the Bernoulli polynomials. We used $A(t) = e^t - 1$, so that $n!P_n(x) = (x + 1)^n - x^n$. Now this equation is also associated with the important set of Newton polynomials

Received February 6, 1939; in preliminary form this paper was presented to the American Mathematical Society, April, 1936 under the title *A characterization of a class of polynomials.*

[1] Acta Mathematica, vol. 48(1926), pp. 279–304.

[2] Trans. Amer. Math. Soc., vol. 39(1936), pp. 345–379, and vol. 41(1937), pp. 153–159.

[3] This Journal, vol. 3(1937), pp. 593–609.

$$N_0(x) = 1, \qquad N_n(x) = \frac{x(x-1)\cdots(x-n+1)}{n!} \qquad (n = 1, 2, \cdots), \tag{1.3}$$

which is *not* an Appell set. Yet, it has properties analogous to those ((1.1) and (1.2)) of Appell polynomials. In fact,

$$\Delta N_n(x) \equiv N_n(x+1) - N_n(x) = N_{n-1}(x), \tag{1.4}$$

$$(1+t)^x = 1 \cdot e^{x \log(1+t)} = \sum_0^\infty N_n(x)t^n. \tag{1.5}$$

It is thus suggested that we define a class of *difference polynomial sets*, of which $\{N_n\}$ is a particular set, by means of the relations

$$\Delta P_n(x) = P_{n-1}(x) \qquad (n = 0, 1, 2, \cdots). \tag{1.6}$$

And more generally, we can use other operators than d/dx and Δ, to define further sets. We thus obtain all polynomial sets of *type zero* (as we denote them). The definition of sets of type zero generalizes readily to give sets of type one, two, $\cdots$, and of infinite type. (This is done immediately after relation (1.15).) This hierarchy of types is all-inclusive, in that every set of polynomials is of a definite type.

The main purpose of this paper is to bring to attention these sets of type zero and to indicate some of their properties. This section considers sets in general. §2 obtains various characterizations of zero type sets. Then, in §3, a study is made of the conditions on a set of zero type in order that it satisfy certain functional equations of finite order. As there are some known Tchebycheff sets that are of type zero, we next (§4) determine all zero type sets that are Tchebycheff sets. Lastly, in §5, we examine some extensions of the definition of type to type of higher order.

By a *set of polynomials* $\{P_n(x)\}$ $(n = 0, 1, 2, \cdots)$ we shall mean a sequence in which each P_n is of degree *exactly* n. We shall denote the set $\{P_n\}$ by P.

LEMMA 1.1. *Let J be a linear operator applicable to the functions x^n $(n = 0, 1, \cdots)$ (and hence to all polynomials) and such that $J[x^n]$ is a polynomial of degree not exceeding n. Then J has the form*

$$J[y(x)] = \sum_0^\infty L_n(x)y^{(n)}(x), \tag{1.7}$$

valid for all polynomials, where $L_n(x)$ is a polynomial of degree not exceeding n.

To see this, define the $L_n(x)$ recurrently by the relations

$$J[x^n] = \sum_{k=0}^n L_k(x)\cdot n(n-1)\cdots(n-k+1)x^{n-k} \qquad (n = 0, 1, \cdots). \tag{1.8}$$

Since the degree of $J[x^n]$ does not exceed n, the degree of the L_n's are seen not to exceed their index. By construction, (1.7) now holds for $y(x) = x^n$, and therefore for all polynomials.

Of special interest is the case where $J[x^n]$ is always of degree $n - 1$.

LEMMA 1.2. *In order that the operator* (1.7) *carry every polynomial into one whose degree is less by precisely*[4] *one, it is necessary and sufficient that*

$$(1.9)\quad L_0(x) = 0, \qquad L_n(x) = l_{n0} + l_{n1}x + \cdots + l_{n,n-1}x^{n-1} \qquad (n = 1, 2, \cdots)$$

and

$$(1.10)\qquad \lambda_n \equiv nl_{10} + n(n-1)l_{21} + \cdots + n!l_{n,n-1} \neq 0 \qquad (n = 1, 2, \cdots).$$

First suppose that $J[1] = 0$ and $J[x^n]$ is a polynomial of degree $n - 1$ $(n = 1, 2, \cdots)$. From (1.8) we find that the coefficients of x^n and x^{n-1} in $J[x^n]$ are respectively

$$l_{00} + nl_{11} + n(n-1)l_{22} + \cdots + n!l_{nn}, \qquad \lambda_n \qquad (n = 0, 1, \cdots).$$

Taking $n = 0, 1, \cdots$ successively, we see that $l_{ii} = 0$ $(i = 0, 1, \cdots)$, so that $L_n(x)$ is of degree less than n; and in order that $L[x^n]$ be of degree exactly $n - 1$, it is necessary that $\lambda_n \neq 0$. The conditions are thus necessary, and it is readily seen that they are also sufficient.

We shall assume without further mention that the operators with which we deal are of type (1.8) and that they fulfill the conditions of Lemma 1.2, so that they have the form

$$(1.11)\qquad J[y] = \sum_{n=1}^{\infty} (l_{n0} + \cdots + l_{n,n-1}x^{n-1})y^{(n)}(x)$$

with $\lambda_n \neq 0$ $(n = 1, 2, \cdots)$.

THEOREM 1.1. *Let P: $\{P_n(x)\}$ be a given set. There is a unique operator J for which*

$$(1.12)\qquad J[P_n] = P_{n-1} \qquad (n = 0, 1, \cdots).$$

If $y(x) = P_n$ $(n = 1, 2, \cdots)$ is substituted into (1.11), it is found that the l_{ij}'s exist to make (1.12) true and are uniquely determined. This is the assertion of the theorem. If P satisfies (1.12) we shall say that P corresponds to the operator J. Conversely, we have

THEOREM 1.2. *To each operator J correspond infinitely many sets P for which* (1.12) *holds. In particular, one and only one of these sets (which we call the basic set and denote by $\{B_n\}$) is such that*[5]

$$(1.13)\qquad B_0(x) = 1; \qquad B_n(0) = 0, \qquad n > 0.$$

If Q is any polynomial of degree s, it is found by direct substitution that a polynomial P exists, unique to within an additive constant, such that $J[P] = Q$;

[4] It is understood that $J[c] = 0$ for every constant c.

[5] The set $\{B_n\}$ is the "best approximation" set relative to the sequence of operators $J^0, J^1, J^2, \cdots$ according to the definition in the American Journal of Mathematics, vol. 57(1935), especially p. 593.

and P is of degree $s + 1$. Choosing $B_0(x) = 1$, we can then successively (and uniquely) determine $B_1, B_2, \cdots$ to satisfy $J[B_n] = B_{n-1}$, $B_n(0) = 0$ $(n > 0)$. Moreover, B_n is of degree exactly n. We thus have the existence of the *basic set*. That infinitely many sets exist is a consequence of the additive constant that is arbitrary. In fact, we have

COROLLARY 1.1. *A necessary and sufficient condition that P be a set corresponding to J is that there exist a sequence of numbers $\{a_n\}$ such that*

$$P_n(x) = a_0B_n(x) + a_1B_{n-1}(x) + \cdots + a_nB_0(x) \qquad (a_0 \neq 0). \tag{1.14}$$

Here the B_n's form the basic set for J.

If P satisfies (1.14), then P_n is of degree n, so that P is a set. Again,

$$J[P_n] = \sum a_iJ[B_{n-i}] = \sum a_iB_{n-i-1},$$

so that $J[P_n] = P_{n-1}$. This proves the sufficiency.

To establish the necessity, we first observe that constants $\{a_{ni}\}$ exist so that

$$P_n(x) = a_{n0}B_n(x) + \cdots + a_{nn}B_0(x).$$

From the relation $J[P_n] = P_{n-1}$ it follows that

$$a_{n0}B_{n-1} + \cdots + a_{n,n-1}B_0 = a_{n-1,0}B_{n-1} + \cdots + a_{n-1,n-1}B_0,$$

so that

$$a_{nj} = a_{n-1,j} \qquad (j = 0, 1, \cdots, n-1).$$

For a fixed j, this says that for every $n \geqq j$, all a_{nj}'s with second index j are equal. It is therefore permissible to drop the first index of each a_{nj}; which means that $\{a_j\}$ exists so that P_n is given by (1.14).

It can likewise be shown that

COROLLARY 1.2. *If P is a set corresponding to J, then a necessary and sufficient condition that $\{Q_n\}$ correspond to J is that constants $\{b_n\}$ exist so that*

$$Q_n = b_0P_n + b_1P_{n-1} + \cdots + b_nP_0. \tag{1.15}$$

DEFINITION. Let J be the (unique) operator corresponding to a given set P. P *is of type* k if in (1.11) no coefficient $L_n(x)$ is of degree exceeding k, but at least one is of degree k. If the degrees of the coefficients $L_n(x)$ are unbounded, then P is of *infinite type.*

From Theorem 1.2 follows

COROLLARY 1.3. *There are infinitely many sets for every type (finite or infinite).*

It is of interest to ask under what alterations, either of the set P itself or of the operator J which defines the type of P, the type is preserved. We consider two simple cases.

(i) Suppose P_n is replaced by c_nP_n, where $c_n \neq 0$ $(n = 0, 1, \cdots)$. Such a transformation does not affect convergence properties of series in these polynomials, but it can very well change the type, as is readily established.

(ii) Let the operator

$$K[y] \equiv k_1 y' + k_2 y'' + \cdots \qquad (k_1 \neq 0) \tag{1.16}$$

be given. The following can be shown (analogously to Theorem 1.1): *If P is any set, there exists a unique operator J_K of form*

$$J_K[y] = \sum_{n=1}^{\infty} (l_{n0} + l_{n1}x + \cdots + l_{n,n-1}x^{n-1})K^n[y] \tag{1.17}$$

(where K^n means $K[K^{n-1}[y]]$), such that

$$J_K[P_n] = P_{n-1}. \tag{1.18}$$

Moreover,

$$\lambda_n \equiv nl_{10}k_1 + n(n-1)l_{21}k_1^2 + \cdots + n!l_{n,n-1}k_1^n \neq 0 \quad (n = 1, 2, \cdots). \tag{1.19}$$

There is also an analogue to Theorem 1.2.

Now suppose that we define the type of a set by the degrees of the polynomials $L_n(x)$ in (1.17). It is seen that no matter what operator K (of form (1.16)) is used, the type of P is the same.

2. **On sets of type zero.** Especially simple and important are sets of type zero. We shall find several characterizations for such sets. It will be convenient to restate the condition for a set of type zero as follows: *P is of type zero if*

$$J[P_n] = P_{n-1} \qquad (n = 0, 1, 2, \cdots), \tag{2.1}$$

where

$$J[y] = c_1 y' + c_2 y'' + c_3 y''' + \cdots \qquad (c_1 \neq 0). \tag{2.2}$$

Definition. The formal series

$$J(t) \cong c_1 t + c_2 t^2 + c_3 t^3 + \cdots \tag{2.3}$$

will be called the *generating series* (*or function*) for the operator (2.2).

That Appell sets are of type zero follows from the fact that the generating series is $J(t) = t$. Similarly, for Newton polynomials (and for all the *difference sets*—cf. (1.6)), $J(t) = e^t - 1$.

Let P be of type zero, corresponding to the operator J. Let the formal power series inverse of (2.3) be

$$H(t) \cong s_1 t + s_2 t^2 + \cdots \qquad (s_1 = c_1^{-1} \neq 0), \tag{2.4}$$

obtained from[6]

$$J(H(t)) \cong H(J(t)) \cong t. \tag{2.5}$$

If e is raised to the power $xH(t)$, the expression will have a formal power series expansion in t, in which the coefficient of t^n involves only $s_1, \cdots, s_n$. On multiplying by the formal power series[7]

$$A(t) \cong \sum_0^\infty a_n t^n \qquad (a_0 \neq 0), \tag{2.6}$$

a new series (in t) is obtained, in which the coefficient of t^n now involves only $a_0, \cdots, a_n; s_1, \cdots, s_n$. In fact, this coefficient is a polynomial in x of degree n, and we have, furthermore,

THEOREM 2.1. *A necessary and sufficient condition that P be of type zero corresponding to the operator J of (2.2) is that $\{a_n\}$ exist so that*

$$A(t)e^{xH(t)} \cong \sum_{n=0}^\infty P_n(x)t^n. \tag{2.7}$$

From (1.14) of Corollary 1.1 it is seen that both the necessary and sufficient parts will follow if we can show that for the basic set $\{B_n\}$ (corresponding to J) we have

$$e^{xH(t)} \cong \sum B_n(x)t^n. \tag{2.8}$$

Let exp $\{xH(t)\}$ have the expansion $\sum C_n(x)t^n$. Then $C_n(x)$ is a polynomial of degree exactly n. On setting $x = 0$, we obtain $1 \cong \sum C_n(0)t^n$, so that $C_0(0) = C_0(x) = 1$, $C_n(0) = 0$ $(n > 0)$. By Theorem 1.2, $\{C_n\}$ will therefore be the basic set if we establish the relation $J[C_n] = C_{n-1}$ $(n = 0, 1, \cdots)$. Operate on the $C_n(x)$-series with J. This gives

$$\sum_0^\infty J[C_n]t^n \cong J[\exp\{xH(t)\}] \cong \{c_1H + c_2H^2 + \cdots\}\cdot\exp\{xH\}$$
$$\cong J(H(t))\cdot\exp\{xH\} \cong t\cdot\exp\{xH\} \cong \sum_0^\infty C_{n-1}(x)t^n;$$

[6] If the series for $J(t)$ is formally substituted for t in (2.4), and coefficients combined (in the usual way) to form a single power series in t, the coefficient of t^n is for each n a polynomial in $c_1, c_2, \cdots, c_n, s_1, \cdots, s_n$. It is possible to choose s_n recurrently and uniquely as a simple function of $c_1, \cdots, c_n, s_1, \cdots, s_{n-1}$, so that the power series reduces to the single term t. This sequence of s_i's is the one to be used in (2.4).

[7] The condition $a_0 \neq 0$ is to insure that $P_n(x)$ in (2.7) is of degree n and not less. But $a_0 \neq 0$ is no essential restriction. See, for example, the footnote on page 916 of Bull. Amer. Math. Soc., vol. 41(1935).

and on comparing like powers of t, we obtain $J[C_n] = C_{n-1}$. Thus, (2.8), holds.[8]

Corollary 2.1. *In* (2.7) *the numbers* $\{a_n\}$ *are the same as in* (1.14).

Thus, for Appell sets, (2.7) holds with $H(t) = t$ (cf. (1.2)), and for difference sets (including the Newton polynomials), $H(t) = \log(1 + t)$, so that *a necessary and sufficient condition for a difference set P is that*

$$A(t)(1+t)^x \cong \sum P_n(x)t^n. \tag{2.9}$$

Another familiar set of polynomials of type zero is given by the Laguerre polynomials which satisfy the relation

$$\frac{1}{1-t}\cdot\exp\left\{\frac{-tx}{1-t}\right\} = \sum L_n(x)t^n. \tag{2.10}$$

For this set,

$$A(t) = \frac{1}{1-t}, \qquad H(t) = J(t) = \frac{-t}{1-t} = -\sum_0^\infty t^{n+1},$$

so that

$$L_{n-1}(x) = -(L_n' + L_n'' + L_n''' + \cdots).$$

[8] Some words are in order regarding the validity of the above proof, which uses *formal series*; particularly, since like arguments (as well as obvious modifications) will be used again. Let k be any positive integer. Consider the operator

$$J_k[y] = c_1y' + \cdots + c_ky^{(k)}$$

and the generating series

$$J_k(t) = c_1t + \cdots + c_kt^k.$$

Let $H_k(t)$ be the inverse of $J_k(t)$:

$$J_k(H_k(t)) = H_k(J_k(t)) = t,$$

and define $C_{kn}(x)$ by the *convergent* expansion

$$\exp\{xH_k(t)\} = \sum_{n=0}^\infty C_{kn}(x)t^n.$$

The argument advanced above is now completely legitimate, giving the relations

$$J_k[C_{kn}(x)] = C_{k,n-1}(x).$$

Now it is readily seen that the series for $H(t)$ and for $H_k(t)$ agree through the term in t^k, whence the same is true for the two series for $\exp\{xH(t)\}$ and $\exp\{xH_k(t)\}$. This means that $C_{kn}(x) \equiv C_n(x)$ ($n = 0, 1, \cdots, k$). As k is arbitrary, it follows that $J[C_n] = C_{n-1}$ for *all n*. This establishes (2.8) and, therefore, (2.7).

Having thus shown that the use of formal power series yields the correct result (in the above case), we shall not hesitate to use such power series in what follows, leaving the argument in the present footnote as a guide to further "validity proofs".

Every set satisfies infinitely many linear functional equations. One of the simplest for sets of zero type is given by

THEOREM 2.2. *Let P be of type zero corresponding to operator J, and let $A(t)$ be its determining series. Then P satisfies the equation*

$$L[y(x)] \equiv \sum_{k=1}^{\infty} (q_{k0} + xq_{k1})J^k[y] = \lambda y, \tag{2.11}$$

where $\lambda = n$ for $y = P_n(x)$. The q's are defined by

$$\frac{A'(t)}{A(t)} \cong \sum_0^{\infty} q_{n+1,0}t^n, \tag{2.12}$$

$$H'(t) \cong \sum_0^{\infty} q_{n+1,1}t^n. \tag{2.13}$$

Suppose each side of (2.11) (with $y = P_n$, $\lambda = n$) is multiplied by t^n and a summation made from $n = 0$ to $n = \infty$. There result two power series in t. (2.11) will be established if we show that these series are formally equal. Now the right-hand series is

$$t \sum nP_n t^{n-1} \cong t\frac{d}{dt}\{Ae^{xH}\} \cong te^{xH}\{A' + xH'A\}.$$

Also,

$$\sum J^k[P_n]t^n \cong t^k \sum P_n t^n \cong t^k A e^{xH},$$

so that the left-hand series is

$$\sum_{n=0}^{\infty} \sum_{k=1}^{\infty} (q_{k0} + xq_{k1})J^k[P_n]t^n \cong \sum_{k=1}^{\infty} (q_{k0} + xq_{k1})t^k Ae^{xH},$$

and if we use (2.12) and (2.13), this becomes

$$Ae^{xH}\left\{t\frac{A'}{A} + xtH'\right\}.$$

Hence, the two series are equal, and (2.11) holds.

Since (2.11) is linear and homogeneous, multiplication of a solution by a constant again yields a solution. But such multiplication may destroy the property of being a zero type set. We cannot therefore obtain a complete converse to Theorem 2.2. But we do have

COROLLARY 2.2. *Given an operator J. If a set P satisfies an equation of form* (2.11), *where $\lambda = n$ for $y = P_n(x)$, and if $\{q_{n1}\}$ is related to J by* (2.13), *then non-zero constants $\{h_n\}$ exist so that $\{h_nP_n\}$ is a set of type zero corresponding to J. Its determining series $A(t)$ is then given by* (2.12).

For, define $A(t)$ by (2.12), the arbitrary multiplicative constant which enters being given any non-zero value. By Theorem 2.2, the set $\{R_n\}$, of type zero, corresponding to J and with determining series $A(t)$, satisfies (2.11). Now, it is readily found that for $\lambda = n$ equation (2.11) has a polynomial solution, and that this polynomial is unique to within an arbitrary multiplicative constant. Hence, $\{h_n\}$ exists so that $R_n = h_n P_n$.

As a characterization of sets of type zero, Theorem 2.2 is not wholly satisfactory, since it involves the operator J of the set. This objection is removed in

THEOREM 2.3. *If P is of type zero, it satisfies an equation of the form*

$$M[y(x)] \equiv \sum_{k=1}^{\infty} (r_{k0} + x r_{k1}) y^{(k)}(x) = \lambda y(x), \tag{2.14}$$

where $\lambda = n$ for $y = P_n$. Moreover, the operator J and determining function A corresponding to P are related to the r's as follows:

$$\sum_{k=1}^{\infty} r_{k0} t^k \cong \frac{uA'(u)}{A(u)}, \tag{2.15}$$

$$\sum_{k=1}^{\infty} r_{k1} t^k \cong uH'(u), \tag{2.16}$$

where $u = J(t)$. Conversely, if a set P satisfies equation (2.14), *then non-zero constants $\{h_n\}$ exist such that $\{h_n P_n\}$ is of type zero.*

To see this we observe that if in (2.11) we write out each $J^k[y]$ as a series of derivatives of y and collect all terms with the same order of derivative, then to each k there are only a finite number of terms in $y^{(k)}(x)$. The result of this collecting of terms is to give us the equivalent equation (2.14).

If in (2.11) and (2.14) we replace each derivative $y^{(k)}(x)$ by t^k, we obtain of course the same formal series (since (2.14) is merely a regrouping of terms in (2.11)). That is,

$$\sum_{k=1}^{\infty} (r_{k0} + x r_{k1}) t^k \cong \sum_{k=1}^{\infty} (q_{k0} + x q_{k1}) J^k(t).$$

On using (2.12) and (2.13), we obtain (2.15) and (2.16). The converse follows as in Corollary 2.2.

One obtains a generalization by replacing $y^{(k)}(x)$ in $M[y]$ by $K^k[y]$, where K is an operator of form (1.16). This yields the following theorem (proved as was Theorem 2.3):

THEOREM 2.4. *Let operator K be given. If P is a set of type zero, it satisfies an equation of form*[9]

$$T[y(x)] \equiv \sum_{k=1}^{\infty} (r_{k0} + x r_{k1}) K^k[y] = \lambda y(x), \tag{2.17}$$

where $\lambda = n$ for $y = P_n$. The operator J and determining function A corresponding to P are related to the r's as follows:

[9] We observe, on comparing the coefficient of x^n on both sides of (2.17), that $r_{11} = k_1^{-1}$.

$$\sum_{k=1}^{\infty} r_{k0}[K(t)]^k \cong \frac{uA'(u)}{A(u)}, \tag{2.18}$$

$$\sum_{k=1}^{\infty} r_{k1}[K(t)]^k \cong uH'(u), \qquad u = J(t). \tag{2.19}$$

Conversely, if a set P satisfies equation (2.17), *then non-zero constants $\{h_n\}$ exist so that $\{h_nP_n\}$ is of type zero.*

From (2.11) of Theorem 2.2 there follows a further characterization of sets of type zero, expressed solely in terms of the members of the set itself. It is

THEOREM 2.5. *A necessary and sufficient condition that a set P be of type zero is that constants q_{k0}, q_{k1} exist so that*

$$\sum_{k=1}^{\infty} (q_{k0} + xq_{k1})P_{n-k}(x) = nP_n(x) \qquad (n = 1, 2, \cdots). \tag{2.20}$$

The operator J and the determining series A for P are related to the q's by (2.12) *and* (2.13).

Let (2.7) be differentiated with respect to x. On equating coefficients of like powers of t, we obtain

$$P_n'(x) = s_1P_{n-1}(x) + s_2P_{n-2}(x) + \cdots + s_nP_0(x) \qquad (n = 1, 2, \cdots), \tag{2.21}$$

whence we have

THEOREM 2.6. *A necessary and sufficient condition that a set P be of type zero is that constants $\{s_n\}$ exist for which* (2.21) *holds; in this case the operator J corresponding to P is determined through $\{s_n\}$ by means of* (2.4) *and* (2.5).[10]

Theorem 2.6 will later be seen to generalize to sets of all types. (Cf. Lemma 5.1.) It is of interest to compare (2.20) and (2.21). The latter involves only J (through H), so that *all* zero type sets for one and the same operator satisfy the *same* equation of form (2.21). On the other hand, both J and A are involved by the constants present in (2.20), so that if sets $\{P_n\}$ and $\{Q_n\}$ both satisfy (2.20), then there is a $c \neq 0$ such that $Q_n = cP_n$, $n > 1$; i.e., there is an essentially unique set satisfying (2.20) for given q_{k0}, q_{k1}.

If (2.7) is differentiated with respect to x, the left side is multiplied by $H(t)$, and P_n' replaces P_n on the right. Recalling that H begins with a term in t, we see that $Q_n = P_{n+1}'$ is a set of zero type, corresponding to the same operator J as does P. In other words, we have

THEOREM 2.7. *If P is of type zero, then so are the sets $\{P_{n+1}'\}$, $\{P_{n+2}''\}$, $\{P_{n+3}'''\}$, $\cdots$; and they all correspond to the same operator as does P. More*

[10] A simple extension of Theorem 2.6 is the following: *Let P be a set with operator J whose inverse is H, and let K be an operator of form* (1.16). *Set*

$$K(H(t)) \cong \alpha_1 t + \alpha_2 t^2 + \cdots .$$

Then

$$K[P_n(x)] = \alpha_1P_{n-1}(x) + \alpha_2P_{n-2}(x) + \cdots + \alpha_nP_0(x) \qquad (n = 1, 2, \cdots)$$

generally, let K be of form (1.16). Then, if P is of type zero, so are $\{K[P_{n+1}]\}$, $\{K^2[P_{n+2}]\}$, $\cdots$, and they correspond to the same operator as does P.

Let us apply the preceding characterizations to some well-known zero-type sets.

Example 1. $P_n(x) = x^n/n!$. Then

$$\sum_0^\infty P_n(x)t^n = e^{tx}, \qquad A(t) = 1, \qquad H(t) = J(t) = t.$$

It is readily determined that (2.11) and (2.14) become

$$xP_n' = nP_n,$$

and (2.20), (2.21) become, respectively,

$$xP_{n-1} = nP_n, \qquad P_n' = P_{n-1}.$$

Also, (2.17) holds with $r_{k0} = 0$ and r_{k1} determined from

$$\sum_{k=1}^\infty r_{k1}[K(t)]^k = t.$$

Example 2. Laguerre polynomials $\{L_n(x)\}$. Here

$$\sum_0^\infty L_n(x)t^n = \left(\frac{1}{1-t}\right)\exp\left\{\frac{-xt}{1-t}\right\}, \qquad A(t) = \frac{1}{1-t}, \qquad J(t) = H(t) = \frac{-t}{1-t}.$$

It is found that (2.11) and (2.14) become

$$\sum_{k=1}^\infty (1 - kx)J^k[L_n(x)] = nL_n(x) \qquad (J[y] \equiv -y' - y'' - y''' - \cdots),$$

$$(x-1)L_n' - xL_n'' = nL_n,$$

while (2.20), (2.21) reduce to

$$\sum_{k=1}^\infty (1 - kx)L_{n-k}(x) = nL_n(x), \qquad L_n'(x) = -[L_{n-1}(x) + L_{n-2}(x) + \cdots].$$

Most of these relations are known. In (2.17), r_{k0}, r_{k1} are determined by the series

$$\sum_{k=1}^\infty r_{k0}[K(t)]^k = -t, \qquad \sum_{k=1}^\infty r_{k1}[K(t)]^k = t - t^2.$$

Example 3. Hermite polynomials $\{H_n(x)\}$. Here[11]

$$\sum_{n=0}^\infty H_n(x)t^n = \exp\{-t^2 + 2tx\}, \qquad A(t) = e^{-t^2}, \qquad H(t) = 2t, \qquad J(t) = \tfrac{1}{2}t.$$

[11] It is more common to define $H_n(x)$ so that $H_n/n!$ is the coefficient of t^n. We find it simpler to adopt the present definition.

Relations (2.11) and (2.14) become

$$2xH'_n - H''_n = 2nH_n ,$$

and (2.20), (2.21) become

$$2xH_{n-1} - 2H_{n-2} = nH_n , \qquad H'_n = 2H_{n-1} .$$

These are also well known. The defining relations for r_{k0} , r_{k1} in (2.17) are

$$\sum_{k=1}^{\infty} r_{k0}[K(t)]^k = -\tfrac{1}{2}t^2, \qquad \sum_{k=1}^{\infty} r_{k1}[K(t)]^k = t.$$

Example 4. Newton Polynomials. Here

$$\sum_0^{\infty} N_n(x)t^n = (1 + t)^x, \qquad A(t) = 1, \qquad H(t) = \log (1 + t), \qquad J(t) = e^t - 1.$$

It is seen that (2.11), (2.14), (2.20), (2.21) become, respectively,

$$x \sum_{k=1}^{\infty} (-1)^{k-1} \Delta^k N_n(x) = nN_n(x), \qquad x \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} N_n^{(k)}(x) = nN_n(x),$$

$$x \sum_{k=1}^{\infty} (-1)^{k-1} N_{n-k}(x) = nN_n(x),$$

$$N'_n(x) = N_{n-1}(x) - \frac{1}{2} N_{n-2}(x) + \cdots + \frac{(-1)^{n-1}}{n} N_0(x).$$

In (2.17), $r_{k0} = 0$ and r_{k1} is determined by

$$\sum_{k=1}^{\infty} r_{k1}[K(t)]^k = 1 - e^{-t}.$$

3. **On sets of zero type satisfying finite order equations.** In the Bulletin paper (cited in footnote 7) those Appell sets were determined that satisfy a finite order linear differential equation with polynomial coefficients. Here we extend the problem to the case of a finite order equation in an operator K of form (1.16), satisfied by a zero type set P corresponding to the operator J and determining function A. We first restrict our attention to equations of form (2.17).

In order that a set P, corresponding to a given J and A, satisfy a finite order equation of form (2.17):

$$T[y(x)] \equiv \sum_{k=1}^{s} (r_{k0} + xr_{k1})K^k[y] = \lambda y(x), \tag{3.1}$$

with $\lambda = n$ for $y = P_n(x)$, it is necessary and sufficient that the following relations hold:[12]

[12] It was shown in a footnote to Theorem 2.4 that $r_{11} = k_1^{-1}$.

$$(3.2)\qquad \sum_{k=1}^{m} r_{k0}[K(t)]^k = F_0(t),$$

$$(3.3)\qquad \sum_{k=1}^{q} r_{k1}[K(t)]^k = F_1(t),$$

where $s = \max(m, q)$ and F_0, F_1 are defined by

$$(3.4)\qquad F_0(t) = \{uA'(u)/A(u)\}, \qquad F_1(t) = \{uH'(u)\} \qquad (u = J(t)).$$

Suppose P satisfies (3.1). Since $r_{11} = 1/k_1 \neq 0$, at least one r_{k1} is not zero, and we let r_{q1} be the last non-zero one. On the other hand, all the r_{k0}'s may be zero. This is true, as we see from (3.2), if and only if $A(t)$ is a constant. (3.3) then gives us

THEOREM 3.1. *If P corresponds to a given J and A, with $A(t) \equiv c \neq 0$, then a necessary and sufficient condition that P satisfy a finite order equation of form* (3.1) *is that $F_1(t)$ be a polynomial in $K(t)$.*

Now suppose that $A \not\equiv c$. Then the r_{k0}'s are not all zero, and we let r_{m0} be the last non-zero one. Since (3.2) and (3.3) are polynomials having a common root $K(t)$, their Sylvester determinant vanishes:

$$(3.5)\qquad \begin{array}{r} q \text{ rows}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \\ m \text{ rows}\left\{\begin{array}{c}\\ \\ \\ \end{array}\right. \end{array} \begin{vmatrix} r_{m0}\; r_{m-1,0} & \cdots & r_{10} - F_0(t) & \cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots & r_{m0}\; r_{m-1,0} & \cdots & & r_{10} - F_0(t) \\ r_{q1}\; r_{q-1,1} & \cdots & r_{11} - F_1(t) & \cdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots & r_{q1}\; r_{q-1,1} & \cdots & & r_{11} - F_1(t) \end{vmatrix} = 0.$$

(In (3.5), elements that are zero are not indicated.)

Conversely, suppose (3.5) holds for a choice of the r's, with $r_{m0} \neq 0$, $r_{q1} \neq 0$, and let $s = \max(m, q)$. Then there is a common solution $K(t)$ of (3.2) and (3.3), which solution can be expressed as a formal power series in t. Now from (1.16), K must begin with a term in t. Suppose the common solution $K(t)$ does not have this property. Then let us set $K(t) = c + K_1(t)$, where $K_1(0) = 0$. On substituting into (3.2) and (3.3), we obtain two polynomials (on the left) with constant terms c_0, c_1, say. Now for $t = 0$, since $F_0(0) = F_1(0) = 0$, we get $c_0 = c_1 = 0$. Hence, $K_1(t)$ is also a common solution, and it is zero at $t = 0$. We shall then use K_1, or, what amounts to the same thing, we may suppose that $K(0) = 0$ to begin with. If K begins with a term of higher degree than one in t, then so does the left side of (3.3), and therefore the right side. But $tH'(t)$ begins with a term in t, as does J; hence, so does $F_1(t)$. As this is a contradiction, it follows that $K(t)$ begins with a term in t, and thus the common solution K has the properties of (1.16). Therefore P satisfies (3.1). This establishes

THEOREM 3.2. *Let P correspond to J and A, where $A \not\equiv c$. A necessary and sufficient condition that P satisfy a finite order equation of form* (3.1) *is that $F_0(t)$ and $F_1(t)$ satisfy* (3.5) *for some choice of r_{k0}, r_{k1} (with $r_{m0} \neq 0$, $r_{q1} \neq 0$).*

As (3.5) is to be an identity in t, it will likewise be an identity in u where $u = J(t)$ (so that $t = H(u)$). Accordingly we have

COROLLARY 3.1. *In Theorem* 3.2 *the functions* $F_0(t)$, $F_1(t)$ *may be replaced by* $uA'(u)/A(u)$, $uH'(u)$.

Theorem 3.1 can be generalized to

THEOREM 3.3. *If in Theorem* 3.1 *the condition* "$A(t) \equiv c \neq 0$" *is replaced by the condition*

$$F_0(t) = \sum_{k=1}^{p} \sigma_p[F_1(t)]^p \qquad (\sigma_p \neq 0), \tag{3.6}$$

then the same conclusion holds.

For, $K(t)$ can be defined by (3.3), whereupon $F_0(t)$ is expressible as a polynomial in $K(t)$. That is, an equation of form (3.2) holds. Hence, P satisfies (3.1).

There are infinitely many pairs of functions F_0, F_1 satisfying a given relation (3.5). One can, for example, give *one* of F_0, F_1 arbitrarily. This suggests examination of the following question: Given one of the three elements J, A, K, to what extent are the others determined so that (3.1) holds?

Case I. *Given* $J(t)$. This determines $H(t)$ and therefore $F_1(t)$. In fact, one easily finds from (2.5) that

$$F_1(t) = J(t) \div J'(t). \tag{3.7}$$

DEFINITION. Given a function (or formal series) $f(t)$, beginning with a term in t. We denote by $\mathfrak{F}\{f(t)\}$ the class of all (formal) series $z(t)$, beginning with a term in t, satisfying a relation of the form

$$p_1 z + p_2 z^2 + \cdots + p_n z^n = f(t), \tag{3.8}$$

the p's being constants, with $p_1 \neq 0$. $\mathfrak{F}\{f(t)\}$ thus represents a special class of algebraic functions of $f(t)$.

In terms of this definition, we see that K is determined from (3.3) as a member of the class $\mathfrak{F}\{F_1(t)\}$. That is, J being given, K must be in $\mathfrak{F}\{F_1(t)\}$, but can be an arbitrary member of this class. Consider any such $K(t)$. From $u = J(t)$ follows $t = H(u)$, so that $K(t) = K(H(u)) = K^*(u)$. $A(u)$ is then determined by the (necessary and sufficient) condition that $uA'(u)/A(u)$ be a polynomial in $K^*(u)$. Observing the wide choice possible for K, after which a further wide choice for A exists, we see that to each operator J correspond a large variety of polynomial sets P satisfying an equation of form (3.1).

Case II. *Given* $K(t)$. Then $F_1(t)$ is to be a polynomial in $K(t)$, lacking a constant term and with coefficient of the linear term equal to k_1^{-1}. For all such F_1, we determine $J(t)$ from (3.7). Having now K and J, we obtain A as in Case I.

Case III. *Given* $A(t)$. Let $K^*(u)$ be any member of the class $\mathfrak{F}\{uA'(u)/A(u)\}$. Determine $uH'(u)$ as any polynomial in $K^*(u)$ beginning with a linear term, and solve for $H(u)$. Relation (2.5) gives us $J(u)$, and with this J we determine $K(t)$ from the identity $K(t) = K^*(J(t))$.

As illustration, choose $K(t) \equiv t$, so that (3.1) is a differential equation. From Case II we see that

$$\frac{J'}{J} = \frac{1}{tQ(t)},$$

where $Q(t)$ is any polynomial of form

$$Q(t) = 1 + l_1 t + \cdots + l_q t^q, \tag{3.9}$$

and that therefore

$$J(t) = ct \cdot \exp\left\{\int_0^t \frac{1 - Q(t)}{tQ(t)}\, dt\right\}. \tag{3.10}$$

The inverse function H can be found from the power series for (3.10) or from the differential equation[13]

$$uH'(u) = H(u)\{1 + l_1 H(u) + \cdots + l_q H^q(u)\}, \tag{3.11}$$

with the condition that $H(u)$ begins with the term t/c. And finally, A is obtained as the solution of the differential equation

$$\frac{uA'(u)}{A(u)} = b_1 H(u) + \cdots + b_m H^m(u),$$

where the b's are arbitrary, but $b_1 \neq 0$; i.e.,

$$A(u) = \gamma \cdot \exp\left\{\int_0^u \frac{1}{u}\,[b_1 H(u) + \cdots + b_m H^m(u)]\, du\right\}, \tag{3.12}$$

γ = arbitrary constant.

Relations (3.10), (3.11) *and* (3.12) *are thus necessary and sufficient conditions that equation* (3.1) *be satisfied for* $K(t) = t$.

It has already been remarked that (2.17) is not the only linear functional equation satisfied by a set P. In fact, extending a result in the Bulletin paper (loc. cit., p. 914), it is easy to show that given an operator K of form (1.16), and given any set P (which need not be of type zero), polynomials $\{L_n(x)\}$ with L_n of degree $\leqq n$, and characteristic numbers $\{\lambda_n\}$, can be chosen, and indeed in infinitely many ways, so that the set P is a solution of the equation[14]

$$L[y(x)] \equiv \sum_{n=1}^{\infty} L_n(x) K^n[y] = \lambda y \tag{3.13}$$

(with $\lambda = \lambda_n$ for $y = P_n$).

[13] $K(t) = t = K^*(u) = K(H(u))$. Therefore, $H(u) = t$, and $K^*(u) = H(u)$. (3.11) then follows from $tQ(t) = uH'(u)$ $(u = J(t))$ if we write $t = H(u)$.

[14] It is no restriction to have the summation begin with $n = 1$, for if a term $n = 0$ is present, it is of the form $L_0 y = cy$, and this can be absorbed into the right side. The only effect is to alter all the λ_n's by the amount $-c$.

An equation (3.13) is said to be of *finite order* r if $L_n(x) \equiv 0$ for $n > r$, but $L_r \not\equiv 0$. We now investigate conditions under which a set P of type zero satisfies a finite order equation of form (3.13).

LEMMA 3.1. *A set P (not necessarily of type zero) cannot satisfy two different equations of form* (3.13) *if the characteristic numbers are respectively the same.*

For, suppose P satisfies (3.13) and also $L^*[y] = \lambda y$ (whose coefficients and characteristic numbers are $L_n^*(x)$ and $\lambda_n^* = \lambda_n$). By subtraction,

$$\sum_{n=1}^{\infty} (L_n - L_n^*)K^n[P_s] = 0 \qquad (s = 1, 2, \cdots).$$

Now, $K^n[P_s] = 0$, $n > s$, and $K^n[P_n] =$ constant $\neq 0$. Hence, on setting $s = 1, 2, \cdots$ successively, we find that $L_s - L_s^* = 0$, $s \geqq 1$. The two equations, supposedly different, are thus identical.

If we set

$$L_n(x) = l_{n0} + \cdots + l_{nn}x^n, \tag{3.14}$$

the characteristic numbers λ_n of (3.13) are given by

$$\lambda_n = nk_1l_{11} + n(n-1)k_1^2l_{22} + \cdots + n!k_1^nl_{nn} \ (n = 0, 1, \cdots). \tag{3.15}$$

This is seen on equating the coefficient of x^n on both sides of $L[P_n] = \lambda_nP_n$.

Suppose P is a set of zero type. We know that it satisfies (2.17), which is a particular case of (3.13):

$$T[P_n] \equiv \sum_{k=1}^{\infty} S_{1k}(x)K^k[P_n] = nP_n, \tag{3.16}$$

where

$$S_{1k}(x) = r_{k0} + xr_{k1}, \qquad r_{11} = 1/k_1 \neq 0. \tag{3.17}$$

Now define operators T_k by

$$T_k[y] \equiv T(T-1)(T-2)\cdots(T-k+1)[y]. \tag{3.18}$$

THEOREM 3.4. *If the zero type set P satisfies* (3.13), *then* (3.13) *can be expressed in the canonical form*

$$L[y] \equiv \sum_{n=1}^{\infty} \alpha_n T_n[y] = \lambda y, \tag{3.19}$$

where T_n is given by (3.18) *and α_n by*

$$\alpha_n = l_{nn}\cdot k_1^n. \tag{3.20}$$

To see this, denote the operator of (3.19) by $L^*[y]$. If in (2.17) $K[y]$ and its iterates are replaced by series in derivatives of y, using (1.16), we can write $T[y]$ as[15]

$$\text{(a)} \qquad T[y] = \sum_{k=1}^{\infty} Q_{1k}(x) y^{(k)}(x),$$

where Q_{1k} is a polynomial of degree $\leqq 1$. Iteration gives

$$\text{(b)} \qquad T^n[y] = \sum_{k=1}^{\infty} Q_{nk}(x) y^{(k)}(x),$$

where Q_{nk} is of degree $\leqq \max(n, k)$. From this it follows that

$$\text{(c)} \qquad T_n[y] = \sum_{k=1}^{\infty} R_{nk}(x) y^{(k)}(x),$$

R_{nk} being of degree $\leqq \max(n, k)$.

On replacing each $y^{(k)}(x)$ by its equivalent as a series of iterates of $K[y]$, (c) becomes[16]

$$\text{(d)} \qquad T_n[y] = \sum_{k=1}^{\infty} S_{nk}(x) K^k[y],$$

where S_{nk} is a polynomial of degree $\leqq \max(n, k)$. Since from (3.16) $T[P_n] = nP_n$, therefore

$$\text{(3.21)} \qquad T_n[P_k] = k(k-1) \cdots (k-n+1)P_k,$$

and in particular,

$$\text{(3.22)} \qquad T_n[P_k] = 0, \qquad n > k.$$

Using (3.22) in (d) for $k = 1, 2, \cdots, n-1$, we find that $S_{nk} = 0$, $k < n$, so that (d) can be written

$$\text{(3.23)} \qquad T_n[y] = \sum_{k=n}^{\infty} S_{nk}(x) K^k[y].$$

If we substitute this expression into $L^*[y]$ (given by (3.19)) and collect like iterates of K, we obtain for L^* the form

$$L^*[y] = \sum_{k=1}^{\infty} \{\alpha_1 S_{1k} + \cdots + \alpha_k S_{kk}\} K^k[y].$$

That is, L^* can be written in the form of (3.13). By Lemma 3.1 it will follow that L^* and L are identical if we show that the respective characteristic numbers are the same. For L the numbers are λ_n, given by (3.15). From (3.21) we

[15] Since $K(t)$ begins with a term in t, and therefore $K^s(t)$ with a term in t^s, the coefficient of any $y^{(k)}(x)$ in (a) is obtained from only a *finite number* of coefficients of (2.17). Hence, the coefficients in (a) are well-determined.

[16] The point of the preceding footnote applies here also.

see that λ_n^* (for L^*) is given by

$$\lambda_n^* = \sum_{k=1}^{\infty} \alpha_k \cdot n(n-1) \cdots (n-k+1),$$

and this is precisely λ_n. The theorem is thus established.

COROLLARY 3.2. *Under the conditions of Theorem* 3.4, *the coefficients* $L_n(x)$ *of* (3.13) *are given by*

$$L_n(x) = \alpha_1 S_{1n}(x) + \cdots + \alpha_n S_{nn}(x) \qquad (n = 1, 2, \cdots). \tag{3.24}$$

To justify the phrase "canonical form", it should be shown that every equation of form (3.19) has a solution of zero type. That is,

THEOREM 3.5. *Let K be an operator of form* (1.16), *and let $T[y]$ be of form*

$$T[y] \equiv \sum_{k=1}^{\infty} (r_{k0} + x r_{k1}) K^k[y]$$

with $r_{11} = 1/k_1 \neq 0$. Then for every choice of α's (not all zero), the equation

$$L[y] \equiv \sum_{n=1}^{\infty} \alpha_n T_n[y] = \lambda y$$

is satisfied by a zero type set.

In fact, $T[y]$ serves to define a zero type set P by virtue of Theorem 2.4 and the relation (2.17). This same set P will clearly satisfy the above equation $L[y] = \lambda y$, and this is what was to be shown.[17]

LEMMA 3.2. *In order that a zero type set P satisfy a finite order equation of form* (3.13) *it is necessary and sufficient that in the canonical form* (3.19) *(into which* (3.13) *can be cast) the following two conditions hold:*

$$\begin{cases} \alpha_n = 0, & n > r; \\ \alpha_1 S_{1n}(x) + \cdots + \alpha_r S_{rn}(x) = 0, & n > r. \end{cases} \tag{3.25}$$

If r is the smallest positive integer for which this is true, the equation is of order r.

For, if (3.13) is of order r, then from (3.20), $\alpha_n = 0$, $n > r$; and from (3.24) the other relation of (3.25) follows. Conversely, suppose (3.25) holds. Then (3.24) yields the relations $L_n(x) = 0$, $n > r$. The assertion as to the order is obvious.

The function $K(t)$ can be expanded in a power series in $J(t)$, where J is the operator for set P:

[17] However, it cannot be asserted here (as was the case in Theorem 2.4) that if Q is any set satisfying $L[Q_n] = \lambda_n Q_n$, then there exist non-zero constants c_n such that $P_n = c_n Q_n$ is of type zero. For, it may now happen that two or more λ's are equal. Suppose $\lambda_m = \lambda_n$, which value we call λ'. Q_m and Q_n are solutions of $L[y] = \lambda' y$, and therefore so is $aQ_m + bQ_n$ for all constants a and b. The argument used in Theorem 2.4 (or rather first used in Corollary 2.2) is thus no longer valid. And it cannot be successfully amended.

$$K(t) \cong r_1 J(t) + r_2 J^2(t) + \cdots . \tag{3.26}$$

Similarly,

$$\begin{cases} K^i(t) \cong r_{ii} J^i(t) + r_{i,i+1} J^{i+1}(t) + \cdots & (i = 1, 2, \cdots), \\ \sum_{n=i}^{\infty} r_{in} t^n \cong \left(\sum_{n=1}^{\infty} r_n t^n \right)^i . \end{cases} \tag{3.27}$$

Recalling that $J[P_n] = P_{n-1}$, we find that

$$\begin{cases} K[P_n] = r_1 P_{n-1} + r_2 P_{n-2} + \cdots , \\ K^2[P_n] = \qquad r_{22} P_{n-2} + r_{23} P_{n-3} + \cdots , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K^i[P_n] = \qquad\qquad r_{ii} P_{n-i} + r_{i,i+1} P_{n-i-1} + \cdots . \end{cases} \tag{3.28}$$

And on using these relations and also (3.21) in (3.23), we obtain the further relations

$$\begin{cases} nP_n = \sum_{j=1}^{n} S_{1j} \{ r_{jj} P_{n-j} + r_{j,j+1} P_{n-j-1} + \cdots + r_{jn} P_0 \} & (n = 1, 2, \cdots); \\ n(n-1)P_n = \sum_{j=2}^{n} S_{2j} \{ r_{jj} P_{n-j} + \cdots + r_{jn} P_0 \} & (n = 2, 3, \cdots); \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ n(n-1) \cdots (n - K + 1)P_n & \\ \qquad = \sum_{j=k}^{n} S_{kj} \{ r_{jj} P_{n-j} + \cdots + r_{jn} P_0 \} & (n = k, k+1, \cdots). \end{cases} \tag{3.29}$$

Equations (3.29) enable us to determine the $S_{ij}(x)$'s in terms of the set P.

Let us define $\Theta(t)$ by

$$\Theta(t) \cong r_1 t + r_2 t^2 + \cdots . \tag{3.30}$$

Comparison with (3.26) shows that

$$\Theta(J(t)) \cong K(t). \tag{3.31}$$

Suppose the first relation of (3.29) is multiplied by t^n and the result summed from $n = 1$ to $n = \infty$. The left side becomes $\sum nP_n t^n$, while the right side is seen to reduce to

$$\left(\sum_0^{\infty} P_n t^n \right) \left\{ S_{11} \sum_1^{\infty} r_{1n} t^n + S_{12} \sum_2^{\infty} r_{2n} t^n + \cdots \right\}.$$

From the definition of r_{ij}, the series in the braces represent $\Theta(t)$, $\Theta^2(t)$, $\cdots$. That is,

$$\sum_{n=1}^{\infty} nP_n t^n \cong \left(\sum_0^{\infty} P_n t^n \right) \{ S_{11}\Theta + S_{12}\Theta^2 + \cdots \}.$$

Now $\sum P_n t^n$ is given by (2.7). We thus have

$$(3.32) \qquad t\frac{\partial}{\partial t}\{A(t)e^{xH(t)}\} \cong A(t)e^{xH(t)}\{S_{11}(x)\Theta(t) + S_{12}(x)\Theta^2(t) + \cdots\}.$$

In similar manner we obtain from the second relation of (3.29),

$$(3.33) \qquad t\frac{\partial^2}{\partial t^2}\{Ae^{xH}\} \cong Ae^{xH}\{S_{22}\Theta^2 + S_{23}\Theta^3 + \cdots\};$$

and from the general relation of (3.29),

$$(3.34) \qquad t^k\frac{\partial^k}{\partial t^k}\{Ae^{xH}\} \cong Ae^{xH}\{S_{kk}\Theta^k + S_{k,k+1}\Theta^{k+1} + \cdots\}.$$

These relations permit us to establish

THEOREM 3.6. *Let P be of type zero, with operator J and determining function A. In order that P satisfy a finite order equation of form* (3.13) *it is necessary and sufficient that constants $\alpha_1, \cdots, \alpha_r$ exist, not all zero, such that the function*

$$(3.35) \qquad Q(t, x) \equiv \frac{e^{-xH(t)}}{A(t)}\left[\alpha_1 t\frac{\partial}{\partial t}\{Ae^{xH}\} + \cdots + \alpha_r t\frac{\partial^r}{\partial t^r}\{Ae^{xH}\}\right],$$

when expressed as a power series in $\Theta(t)$, reduces to a polynomial in $\Theta(t)$.

Q can be written as a power series in t, and can therefore (formally) be expressed as a series of powers of $\Theta(t)$. More precisely, from (3.32) to (3.34) we have

$$(3.36) \qquad \begin{aligned} Q(t, x) \cong \{\alpha_1 S_{11}\}\Theta(t) + \cdots + \{\alpha_1 S_{1r} + \cdots + \alpha_r S_{rr}\}\Theta^r(t) \\ + \sum_{n=r+1}^{\infty} \{\alpha_1 S_{1n} + \cdots + \alpha_r S_{rn}\}\Theta^n(t). \end{aligned}$$

Suppose P satisfies a finite order equation, so that conditions (3.25) hold for some r. Then $Q(t, x)$, as seen by (3.36), reduces to a polynomial in $\Theta(t)$. The necessity of Theorem 3.6 is thus proved. Conversely, suppose that for some r $Q(t, x)$ is a polynomial in $\Theta(t)$. We wish to show that set P, corresponding to the A and J in terms of which Q is defined, satisfies a finite order equation. We know that P satisfies (2.17). Using operator T of (2.17), we form the equation

$$(3.37) \qquad L[y] \equiv \sum_{k=1}^{r} \alpha_k T_k[y] = \lambda y,$$

which is also satisfied by P. It is this equation that we shall prove is of finite order.

If $L[y]$ is recast in terms of $K[y]$, so that it is of form (3.13), the coefficients $L_n(x)$ are given by (3.24), with $\alpha_n = 0$ for $n > r$. Now from (3.36), since Q is a polynomial in $\Theta(t)$, there is an integer s such that

$$\alpha_1 S_{1n} + \cdots + \alpha_r S_{rn} = 0, \qquad n > s.$$

Hence $L_n(x) = 0$ for $n > s$. This establishes the sufficiency.

The condition of Theorem 3.6 is more serviceable than is that of Lemma 3.2, since the function Q is determined directly from A and H. We can rid ourselves of the function $\Theta(t)$ on making use of (3.31). It gives us

COROLLARY 3.3. *The zero type set P satisfies a finite order equation if and only if constants $\alpha_1, \cdots, \alpha_r$ (not all zero) exist so that $Q(J(t), x)$ is a polynomial in the function $K(t)$.*

Whenever the choice $r = 1$ is permissible, the condition of Theorem 3.6 (or of Corollary 3.3) is seen to reduce to the conditions (3.2), (3.3) already met.

COROLLARY 3.4. *If P satisfies an r-th order equation* (3.13), *then for this equation Q is given by*

$$Q(t, x) = \sum_{j=1}^{r} L_j(x)\Theta^j(t). \tag{3.38}$$

This follows from (3.36).

Corollary 3.4 enables us to show that neither the Legendre set $\{X_n(x)\}$ nor any set $\{c_nX_n\}$ is of type zero. The Legendre polynomials are given by

$$(1 - 2tx + t^2)^{-\frac{1}{2}} = \sum_0^\infty X_n(x)t^n.$$

If $\{X_n\}$ is of type zero, then the left member is of the form $\exp \{xH(t)\}$. This is readily seen to be impossible.

Now suppose $\{P_n = c_nX_n\}$ $(c_n \neq 0)$ is of type zero. X_n, and therefore P_n, satisfies the finite order equation

$$(1 - x^2)y'' - 2xy' = \lambda y$$

with $\lambda = -n(n + 1)$ for $y = P_n$. Here the operator $K[y]$ is merely $y'(x)$, so that $\Theta(t) = H(t)$. Also, $L_1 = -2x, L_2 = 1 - x^2$. Hence, from the corollary,

$$Q(t, x) = L_1H + L_2H^2 = -2xH + (1 - x^2)H^2.$$

If we equate coefficients of like powers of x on both sides, we get from the x^2 terms: $t^2H'^2 = H^2$, so that $H = ct$; and on using this result in the equation obtained from the x terms, we find that $A(t) \equiv$ constant. Finally, the constant terms tell us that $H \equiv 0$ so that $c = 0$. Hence, $\sum P_nt^n$ has for sum a constant. This contradiction shows that $\{P_n = c_nX_n\}$ is not of type zero.

4. **Zero type sets that are Tchebycheff sets.** The Hermite polynomials are Appell polynomials, and are thus of zero type. They are also Tchebycheff orthogonal polynomials.[18] Another orthogonal set of zero type is the Laguerre

[18] The definition of $H_n(x)$ in Example 3 of §2 requires modification in order to satisfy the condition $H_n'(x) = H_{n-1}(x)$ for an Appell set. But such alteration consists only in multiplying each H_n by a suitable non-zero constant c_n. This being done, it is known that the Hermite set is essentially the only Tchebycheff set that is also an Appell set.

set (cf. (2.10)). This suggests the problem of determining all zero type sets that are orthogonal.

J. Meixner[19] has treated this problem by the use of the Laplace transformation, taking (essentially) the relation (2.7) as the definition of the polynomials under consideration. It is possible to give a quite different treatment by means of the known properties of zero type sets, and this we do here.

As a characterization of an orthogonal set $\{Q_n\}$ we take the relation[20]

$$Q_n(x) = (x + \lambda_n)Q_{n-1}(x) + \mu_n Q_{n-2}(x) \qquad (n = 1, 2, \cdots), \tag{4.1}$$

λ_n, μ_n being real constants with $\mu_n \neq 0$, $n > 1$. If $\{Q_n\}$ is an orthogonal set, so is $\{c_n Q_n\}$, $c_n \neq 0$ (although the multipliers c_n can spoil normality if Q_n has this latter property). We shall therefore set the problem as follows: *For what sets $\{Q_n\}$ satisfying* (4.1) *do there exist non-zero constants c_n such that*

$$P_n(x) = c_n Q_n(x) \qquad (n = 0, 1, \cdots) \tag{4.2}$$

is a set of type[21] *zero?*

Suppose that $\{P_n\}$ of (4.2) is of type zero. From (4.1) we obtain an expression for $nP_n(x)$. Comparing this with the value of $nP_n(x)$ as given by (2.20), we obtain (on equating coefficients of like powers of x):

$$n!c_n = c_0 q_{11}^n, \qquad q_{11}^2\lambda_n = q_{10}q_{21} + (n-1)q_{21}, \tag{4.3}$$

$$q_{11}^4\mu_{n+1} = n\{q_{20}q_{11}^2 - q_{10}q_{11}q_{21} - (q_{21}^2 - q_{11}q_{31})(n-1)\}. \tag{4.4}$$

That is, λ_n is at most linear and μ_n at most quadratic, in n, and μ_n has a factor $(n - 1)$.

[19] *Orthogonale Polynomsysteme mit einer besonderern Gestalt der erzeugenden Funktion,* Journal of London Math. Soc., vol. 9 (1934), pp. 6–13.

[20] As justification we observe first that every set orthogonal according to the classical definition satisfies a relation of form (4.1); and secondly, that Shohat has shown that a necessary and sufficient condition that a set $\{Q_n\}$ (normalized so that the x^n term has a coefficient unity) be orthogonal with respect to a weight function $\psi(x)$ of bounded variation in $(-\infty, +\infty)$ is that $\{Q_n\}$ satisfy (4.1) with $\mu_n \neq 0$ for all $n > 1$. (J. Shohat, Comptes Rendus, vol. 207(1938), pp. 556–558.)

We note that in the Shohat definition of orthogonality it is tacitly assumed that no member of an orthogonal set is orthogonal to itself (relative to the given weight function ψ). It is easy to show that if an "orthogonal" set satisfies (4.1) with $\mu_n = 0$ for some $n > 1$, then at least one polynomial of the set is self-orthogonal. Thus, for example, the set $\{x^n\}$ is "orthogonal" for the following choice of ψ:

$$\psi(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$$

This set also satisfies (4.1) with $\lambda_n = \mu_n = 0$ for all n. This apparent contradiction to the theorem of Shohat is resolved when we note that x^n is orthogonal to itself for every $n > 0$.

A colleague, H. L. Krall, has made the further observation that every set P for which $P_n(0) = 0$ $(n > 0)$ is "orthogonal" relative to the same function ψ above.

[21] We shall also assume for convenience that $P_0(x) = 1$. This is no essential restriction since it means only that all the polynomials $P_n(x)$ are multiplied by *one and the same* non-zero constant. The property of being of zero type is thus unaltered.

This condition is also sufficient. For suppose λ_n, μ_n have this form, and let $\{Q_n\}$ be defined by (4.1). Let $\alpha \neq 0$ be any number and set

$$c_n = \alpha^n/n! \qquad (n = 0, 1, \cdots). \tag{4.5}$$

Now define $\{P_n\}$ by (4.2). We are to show that P is of type zero. From (4.1) and (4.2) follows a relation of form

$$nP_n = (\alpha x + \beta + n\gamma)P_{n-1} + (\delta + n\epsilon)P_{n-2}, \tag{4.6}$$

$\alpha \neq 0$, $\delta + n\epsilon \neq 0$ $(n > 1)$. From this we obtain xP_{n-1} as a linear combination of P_n, P_{n-1}, P_{n-2}. It is now a straightforward matter to show that constants q_{k0}, q_{k1} exist so that

$$T_n \equiv (q_{10} + xq_{11})P_{n-1} + (q_{20} + xq_{21})P_{n-2} + \cdots$$

is identically equal to nP_n. They are, in fact, determined by the relations

$$q_{k+2,1} = \gamma q_{k+1,1} + \epsilon q_{k1}, \tag{4.7}$$

$$q_{k+1,0} = \frac{1}{\alpha}\{q_{k1}(\delta - (k-1)\epsilon) + q_{k+1,1}(\beta - \gamma k) + (k+1)q_{k+2,1}\}. \tag{4.8}$$

Thus (2.20) holds, and P is of type zero. That is, we can state

THEOREM 4.1. *A necessary and sufficient condition that an orthogonal set* $\{Q_n\}$' *given by* (4.1), *be such that* $P_n = c_nQ_n$ *is of type zero for some choice of* $c_n \neq 0$ *is that* λ_n, μ_n *have the form*

$$\lambda_n = a + bn, \qquad \mu_n = (n-1)(c + dn), \tag{4.9}$$

with $c + dn \neq 0$ *for* $n > 1$.

As it stands this criterion does not reveal the sets P that are both orthogonal and of type zero. We therefore examine the problem more closely. Relation (4.8) shows that $\{q_{k0}\}$ is determined when $\{q_{k1}\}$ is known. Let us then turn to the recurrence relation (4.7). The characteristic equation is

$$u^2 - \gamma u - \epsilon = 0. \tag{4.10}$$

Case I. $\gamma^2 + 4\epsilon = 0$. Using the initial conditions $q_{11} = \alpha$, $q_{21} = \alpha\gamma$, we obtain

$$q_{k1} = \alpha k(\tfrac{1}{2}\gamma)^{k-1}, \qquad q_{k+1,0} = (\tfrac{1}{2}\gamma)^{k-1}\{\tfrac{1}{2}(\beta\gamma + 2\delta)k + \tfrac{1}{2}\gamma(\beta + \gamma)\}. \tag{4.11}$$

Then from (2.12) and (2.13), we get[22]

[22] The presence of the parameter μ removes the earlier condition that $P_0(x) = 1$. It should also be noticed that we must have $\gamma \neq 0$ in (4.18). The case $\gamma = 0$ is special. For if $\gamma = 0$, then $\epsilon = 0$, and

$$q_{11} = \alpha, \quad q_{k1} = 0\ (k > 1); \quad q_{10} = \beta, \quad q_{20} = \delta, \quad q_{k0} = 0\ (k > 2).$$

Hence

$$H(t) = \alpha t, \quad J(t) = t/\alpha, \quad A(t) = \mu\cdot\exp\{\beta t + \tfrac{1}{2}\delta t^2\},$$

where $\delta \neq 0$ in order that the condition $\mu_n \neq 0$ be fulfilled.

$$(4.12)\qquad H(t) = \frac{2\alpha t}{2 - \gamma t}, \qquad J(t) = \frac{2t}{2\alpha + \gamma t},$$
$$A(t) = \mu\left(\frac{2 - \gamma t}{2}\right)^{\omega} \cdot \exp\left\{\frac{4(\beta\gamma + 2\delta)}{\gamma^2(2 - \gamma t)}\right\},$$

where

$$\omega \equiv \frac{-2}{\gamma^2}(\gamma^2 - 2\delta) \neq \text{a non-negative integer}$$

(in order that $\mu_n \neq 0$).

Case II. $\gamma^2 + 4\epsilon \neq 0$. Let u_1, u_2 be the roots of (4.10). Then

$$(4.13)\qquad q_{k1} = \alpha(\gamma^2 + 4\epsilon)^{-\frac{1}{2}}\{u_1^k - u_2^k\}, \qquad q_{k+1,0} = (\gamma^2 + 4\epsilon)^{-\frac{1}{2}}\{\lambda u_1^k - \sigma u_2^k\},$$

where λ, σ are constants whose values are readily obtained. Consequently

$$(4.14)\qquad H(t) = \alpha \int_0^t \frac{dt}{1 - \gamma t - \epsilon t^2}.$$

Case II_1. $\epsilon = 0$ (so that $\gamma \neq 0$). Then

$$(4.15)\qquad \begin{cases} H(t) = \dfrac{-\alpha}{\gamma}\log(1 - \gamma t), \qquad J(t) = \dfrac{1}{\gamma}\left(1 - \exp\left\{-\dfrac{\gamma t}{\alpha}\right\}\right), \\ A(t) = \mu(1 - \gamma t)^{-\theta} \cdot \exp\left\{-\dfrac{\delta t}{\gamma}\right\}, \end{cases}$$

where

$$\theta = \frac{1}{\gamma^2}(\beta\gamma + \gamma^2 + \delta),$$

and where $\delta \neq 0$ in order that $\mu_n \neq 0$.

Case II_2. $\epsilon \neq 0$. Let $r_i = 1/u_i$. Then

$$(4.16)\qquad H(t) = \frac{1}{\rho}\log\left\{\frac{r_1(t - r_2)}{r_2(t - r_1)}\right\}, \quad J(t) = \frac{e^{\rho t} - 1}{u_1 e^{\rho t} - u_2}, \quad A(t) = \mu \cdot \frac{(1 - u_2 t)^{h_2}}{(1 - u_1 t)^{h_1}},$$

where

$$(4.17)\qquad \rho = \frac{1}{\alpha}(\gamma^2 + 4\epsilon)^{\frac{1}{2}}, \qquad h_i = (\gamma^2 + 4\epsilon)^{-\frac{1}{2}}\frac{u_i(\beta + \gamma) + (\delta + 2\epsilon)}{u_i}.$$

It is to be noted that in all cases H and J do not involve the parameters β, δ, μ and A does not involve α. It follows that *all sets satisfying a relation of form* (4.6) *and having the same* α, γ, ϵ *correspond to the same operator* J; *and all sets satisfying* (4.6) *with the same* β, γ, δ, ϵ *have the same determining*[23] *function* A.

The many relations obtained for H, J, A involve the original parameters $\alpha, \cdots, \epsilon$ and μ, sometimes in complicated manner. This suggests the possi-

[23] At least to within a constant multiplier (because of the presence of μ).

bility of simplifying by introducing independent combinations of the original parameters as new parameters. In fact, the following relations summarize the various cases. They are, in order: Cases I, I special,[24] II_1, II_2; that is,

$$(4.18) \qquad H(t) = \frac{at}{1 - bt}, \qquad J = \frac{t}{a + bt}, \qquad A = \mu(1 - bt)^c \cdot \exp\left\{\frac{d}{1 - bt}\right\},$$

where a, b, c, d, μ are arbitrary, but $abc\mu \neq 0$.

$$(4.19) \qquad H = at, \qquad J = \frac{t}{a}, \qquad A = \mu \exp\{bt + ct^2\},$$

a, b, c, μ arbitrary, but $ac\mu \neq 0$.

$$(4.20) \qquad H = a \log(1 - bt), \qquad J = \frac{1}{b}\{1 - e^{t/a}\}, \qquad A = \mu e^{ct} \cdot (1 - bt)^d,$$

a, b, c, d, μ arbitrary, but $abc\mu \neq 0$.

$$(4.21)\ H = \frac{1}{a}\log\left\{\frac{b(t - c)}{c(t - b)}\right\}, \quad J = bc\left(\frac{e^{at} - 1}{ce^{at} - b}\right), \quad A = \mu\left(1 - \frac{t}{c}\right)^{d_1} \cdot \left(1 - \frac{t}{b}\right)^{d_2},$$

a, b, c, d_1, d_2, μ arbitrary, but $abc\mu \neq 0$ and[25] $b \neq c$.

We therefore have

THEOREM 4.2. *Let P be of zero type, with operator J and determining function A, so that* (2.7) *holds. A necessary and sufficient condition that P be an orthogonal set is that J, H, A satisfy one of the conditions* (4.18) *to* (4.21).

COROLLARY 4.1. *According to the case, the function Ae^{xH} of* (2.7) *assumes the form:*[26]

$$(4.22) \qquad Ae^{xH} = \mu(1 - bt)^c \cdot \exp\left\{\frac{d + atx}{1 - bt}\right\} \qquad (abc\mu \neq 0),$$

$$(4.23) \qquad Ae^{xH} = \mu \cdot \exp\{t(b + ax) + ct^2\} \qquad (ac\mu \neq 0),$$

$$(4.24) \qquad Ae^{xH} = \mu e^{ct} \cdot (1 - bt)^{d + ax} \qquad (abc\mu \neq 0),$$

$$(4.25) \qquad Ae^{xH} = \mu\left(1 - \frac{t}{c}\right)^{d_1 + x/a} \cdot \left(1 - \frac{t}{b}\right)^{d_2 - x/a} \qquad (abc\mu \neq 0,\ b \neq c).$$

The Laguerre set is a particular case of (4.22) and the Hermite set of (4.23). If it were permissible to choose $c = 0$ in (4.23) and (4.24) we would have as particular cases, respectively, $\{x^n/n!\}$ and the Newton set. Hence, these two sets, while not Tchebycheff sets, are nevertheless limiting sets of Tchebycheff sets.

If we form the functions $\{uA'(u)/A(u)\}$, $\{uH'(u)\}$, evaluated for $u = J(t)$, we find in the respective cases that

[24] Case I special refers to an earlier footnote under Case I.

[25] Also, the condition $\delta + \epsilon n \neq 0$ $(n > 1)$ is to be translated in terms of the present parameters.

[26] (4.25) is subject to the restriction mentioned in the preceding footnote.

$$(4.26)\ \left\{\frac{uA'}{A}\right\} = \frac{bt}{a^2}[a(d-c)+bdt], \qquad \{uH'\} = \frac{t}{a}(a+bt);$$

$$(4.27)\ \left\{\frac{uA'}{A}\right\} = \frac{bt}{a} + \frac{2ct^2}{a^2}, \qquad \{uH'\} = t;$$

$$(4.28)\ \left\{\frac{uA'}{A}\right\} = (1-e^{t/a})(ce^{t/a}-bd) \div be^{t/a}, \qquad \{uH'\} = a(e^{t/a}-1)e^{-t/a};$$

$$(4.29)\ \left\{\frac{uA'}{A}\right\} = \frac{bc(1-e^{at})}{(c-b)}\left[\frac{d_1}{ce^{at}} + \frac{d_2}{b}\right], \qquad \{uH'\} = \frac{(e^{at}-1)(ce^{at}-b)}{a(c-b)e^{at}}.$$

Since in (4.26) and (4.27) the expressions are polynomials, it follows from Theorem 2.3 that the sets P of the first two cases satisfy the respective finite order equations

$$(4.30)\quad M[y(x)] \equiv \left\{\frac{b}{a}(d-c)+x\right\}y'(x) + \left\{\frac{b^2d}{a^2}+\frac{b}{a}x\right\}y''(x) = \lambda y(x),$$

$$(4.31)\quad M[y(x)] \equiv \left\{\frac{b}{a}+x\right\}y'(x) + \frac{2c}{a^2}y''(x) = \lambda y(x),$$

where $\lambda = n$ for $y = P_n$. The sets for the last two cases clearly do not satisfy a finite order equation of form (2.14).

5. **Sets of higher type.** Although the present paper has as its main purpose the treatment of zero type sets, we propose in this section to indicate some extensions to sets of higher type. The definition of higher type depends on what characterization of zero type sets one wishes to generalize. We have given one definition in §1. This we shall call A-type. Thus: *A set P is of* A-*type k if in* (1.12) *the maximum degree of the coefficients $L_n(x)$ is k. If the L_n's are of unbounded degree, P is of infinite* A-*type.*

Let P be an arbitrary set. There exists a unique sequence of formal power series $\{M_n(t)\}$ of form

$$(5.1)\qquad M_n(t) \cong m_{nn}t^n + m_{n,n+1}t^{n+1} + \cdots \qquad (m_{nn} \neq 0)$$

such that[27]

$$(5.2)\qquad e^{tx} \cong \sum_{n=0}^{\infty} P_n(x)M_n(t).$$

DEFINITION. We shall term the set of series (or functions) M: $\{M_n(t)\}$ *the E-associate of set P.*

COROLLARY 5.1. *Let P be a set and M its E-associate. A necessary and sufficient condition that P be of type zero is that formal series*

[27] For our purpose it is a matter of indifference whether or not the series symbolized by $M_n(t)$ converge. (5.1) is regarded as a "carrier" for the coefficients m_{nk}, and (5.2) is merely a concise way of writing infinitely many linear equations in these coefficients.

$$(5.3) \qquad A(t) \cong \sum_0^\infty a_n t^n \quad (a_0 \neq 0), \qquad J(t) \cong \sum_1^\infty c_n t^n \quad (c_1 \neq 0)$$

exist so that

$$(5.4) \qquad M_n(t) = \frac{[J(t)]^n}{A(J(t))} = \left\{\frac{u^n}{A(u)}\right\} \qquad (u = J(t)).$$

For, if P is of type zero, (2.7) holds. And from (5.2), on setting $t = H(u)$,

$$A(u)e^{xH(u)} \cong \sum P_n(x)M_n(H(u))A(u),$$

so that

$$A(u)M_n(H(u)) = u^n.$$

(5.4) follows on inverting: $u = J(t)$.

Conversely, suppose (5.3) and (5.4) are satisfied. Define the set Q to be of type zero, corresponding to operator J and determining function A, and let M^* be its E-associate. Then from the half already established, M_n^* has the value given by (5.4). That is, $M_n^*(t) = M_n(t)$ for all n. But just as M is uniquely determined from knowledge of P and (5.2), so is P uniquely defined by (5.2) when M is given. Hence, Q and P are identical, and P is of type zero.

We now characterize sets of A-type k. Let P be such a set. Then, as we know,

$$(5.5) \qquad L[P_n] = P_{n-1},$$

where

$$(5.6) \qquad L[y(x)] \equiv J_0[y] + xJ_1[y] + \cdots + x^k J_k[y],$$

$J_0, \cdots, J_k$ being linear differential operators with constant coefficients such that

$$(5.7) \qquad J_i(t) \cong a_{i,i+1}t^{i+1} + a_{i,i+2}t^{i+2} + \cdots, \qquad 0 \leqq i \leqq k.$$

Also, in order to insure that L carries every polynomial into another of degree one less (since $L[P_n] = P_{n-1}$), we have the further condition

$$(5.8) \qquad \xi_n = a_{01} + na_{12} + n(n-1)a_{23} + \cdots + n \cdots (n-k+1)a_{k,k+1} \neq 0 \qquad (n = 0, 1, \cdots).$$

From (5.6) and (5.2) we have

$$(a) \qquad \begin{aligned} L[e^{tx}] &\cong \{J_0(t) + xJ_1(t) + \cdots + x^k J_k(t)\}e^{tx} \\ &\cong \sum_0^\infty L[P_n]M_n(t) \cong \sum_0^\infty P_n(x)M_{n+1}(t). \end{aligned}$$

But also, on differentiating (5.2) k times in t, we get

$$\text{(b)} \qquad \{J_0 + \cdots + x^k J_k\} e^{tx} \cong \sum_0^\infty P_n(x) \{J_0 M_n + J_1 M_n' + \cdots + J_k M_n^{(k)}\}.$$

Again, in (a) and (b) both the coefficients of $P_n(x)$ are power series beginning with a term in t^{n+1} (cf. relation (5.8)). These coefficients must therefore be identical. That is,

$$(5.9) \quad M_{n+1}(t) = J_0(t) M_n(t) + J_1(t) M_n'(t) + \cdots + J_k(t) M_n^{(k)}(t) \quad (n = 0, 1, \cdots).$$

This proves the necessary part of

THEOREM 5.1. *Let P be a set and M its E-associate. A necessary and sufficient condition that P be of finite* A*-type is that for some k there exist formal power series* (5.7) *satisfying condition* (5.8) *such that M satisfies* (5.9). *Moreover, if $J_k(t) \not\equiv 0$, P is of* A*-type k.*

The sufficiency is established as follows: (5.7) and (5.8) determine an operator L of form (5.6). The first part of (a) holds to give

$$\text{(c)} \qquad L[e^{tx}] \cong \sum_0^\infty L[P_n] M_n \cong \sum_0^\infty L[P_{n+1}] M_{n+1},$$

while (b) continues to hold. On using (5.9), (b) reduces to

$$\text{(d)} \qquad L[e^{tx}] \cong \sum_0^\infty P_n M_{n+1},$$

so that

$$\text{(e)} \qquad \sum_0^\infty \{L[P_{n+1}] - P_n\} M_{n+1}(t) = 0.$$

This formal identity means that if we rearrange in a power series in t, all coefficients must vanish. Recalling the form (5.1) of the functions M_n, we see that all the braces likewise vanish. That is, $L[P_n] = P_{n-1}$, $n \geqq 1$. That $L[P_0] = 0$ is immediate. Hence P is of A-type $\leqq k$. The statement in the theorem concerning the precise type number is evident.

If $k = 0$, (5.9) reduces to $M_{n+1} = JM_n$ (dropping the subscript 0); i.e., $M_n(t) = M_0(t)[J(t)]^n$. This coincides with the previously found condition (5.4) if $A(t)$ is chosen so that $M_0(t) A(J(t)) = 1$.

It has already been observed that if M is any sequence of power series of form (5.1) (which includes the condition $m_{nn} \neq 0$), then (5.2) uniquely defines a set P for which M is the E-associate. To say that M is the E-associate of some set P is then equivalent to saying that M is of form (5.1).

From Theorem 5.1 we can prove

THEOREM 5.2. *Let M be of form* (5.1), *so that M is the E-associate of a set P. A necessary and sufficient condition that P be of finite* A*-type is that k exist so that the following* $(k + 1)$ *ratios of determinants are independent of n:*

$$
(5.10)\qquad \Delta_j(t) = \begin{vmatrix} M_n & \cdots & M_n^{(j-1)} & M_{n+1} & M_n^{(j+1)} & \cdots & M_n^{(k)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{n+k} & \cdots & M_{n+k}^{(j-1)} & M_{n+k+1} & M_{n+k}^{(j+1)} & \cdots & M_{n+k}^{(k)} \end{vmatrix} \div \begin{vmatrix} M_n & M_n' & \cdots & M_n^{(k)} \\ \cdots & \cdots & \cdots & \cdots \\ M_{n+k} & M_{n+k}' & \cdots & M_{n+k}^{(k)} \end{vmatrix} \quad (j = 0, 1, \cdots, k).
$$

And if $\Delta_k(t) \not\equiv 0$, the A-type is precisely k.

First, suppose P is of A-type k. If in (5.9) we replace n by n, $n + 1$, $\cdots$, $n + k$ respectively, we obtain $k + 1$ equations for $J_0, \cdots, J_k$ whose solution is given by the determinant ratios[28] in (5.10); i.e., $J_j(t) = \Delta_j(t)$. Thus the condition is necessary. Conversely, suppose (5.10) holds, the Δ_j's being independent of n. Then (5.9) is satisfied by the choice $J_j = \Delta_j$. The sufficiency will now follow from Theorem 5.1 if we show that J_j satisfies conditions (5.7) and (5.8).

Suppose the series (5.1) is substituted into (5.10). It is found that the numerator and denominator have as lowest terms, respectively,

$$
\alpha_{jn} t^{nk+j+1} \cdot \prod_{i=n}^{n+k} m_{ii}, \qquad \beta_n t^{nk} \cdot \prod m_{ii},
$$

where

$$
\beta_n = \begin{vmatrix} 1 & n & \cdots & n(n-1)\cdots(n-k+1) \\ 1 & (n+1) & \cdots & (n+1)\cdots(n-k+2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (n+k) & \cdots & (n+k)\cdots(n+1) \end{vmatrix},
$$

and where α_{jn} is obtained by replacing the $(j + 1)$-th column of β_n by the elements $m_{i+n,i+n} \div m_{i+n-1,i+n-1}$ ($i = 1, 2, \cdots, k + 1$). Hence J_j cannot begin with a term of degree less than t^{j+1} if we show that $\beta_n \neq 0$. If in β_n we subtract each row from the one following, we obtain a new determinant (of value β_n), which is $k!$ times β_n', where β_n' is obtained from β_n by changing k to $k - 1$. Working down to $k = 1$ we get $\beta_n = k!(k - 1)! \cdots 2!\,1!$, and this is not equal to zero.

Thus, condition (5.7) holds. There remains to establish (5.8). Since $\Delta_j = J_j$ begins with the term $t^{j+1}\alpha_{jn} \div \beta_n$ (or a term of even higher degree), we see on comparing with (5.7) that $\alpha_{j,j+1} = \alpha_{jn} \div \beta_n$. Hence $\alpha_{j,j+1}$ ($j = 0, 1, \cdots, k$) is the solution of a system of $(k + 1)$ linear non-homogeneous equations, the matrix of whose coefficients is given by the elements of the determinant β_n, and whose non-homogeneous terms are the quantities $m_{i+n,i+n} \div m_{i+n-1,i+n-1}$. The first of these equations is

$$
a_{01} + na_{12} + \cdots + n(n-1)\cdots(n-k+1)a_{k,k+1} = \frac{m_{n+1,n+1}}{m_{nn}}.
$$

[28] That the denominator does not vanish identically is demonstrated later in the proof, where it is shown that $\beta_n \neq 0$.

The left side is the quantity ξ_n of (5.8). As $m_{ii} \neq 0$ for all i, so is $\xi_n \neq 0$ for all n. Thus (5.8) holds and the proof is complete.

We come now to a second definition of type.

LEMMA 5.1. *To each set P corresponds a unique sequence of polynomials $\{T_n(x)\}$, with $T_n(x)$ of degree not exceeding n, such that*[29]

$$P_n' = T_0P_{n-1} + T_1P_{n-2} + \cdots + T_{n-1}P_0 \quad (n = 1, 2, \cdots). \tag{5.11}$$

This is seen if we set $n = 1, 2, \cdots$ successively. It is to be observed that the T_n's do not determine the P_n's uniquely. In fact, there are infinitely many sets satisfying (5.11) for a given sequence $\{T_n\}$.

DEFINITION. *A set P is of* B-*type k if in* (5.11) *the maximum degree of the polynomials $T_n(x)$ is k. Otherwise, P is of infinite* B-*type.*

If P is of B-type zero, the T_n's are constants, so that (5.11) reduces to (2.21). That is, P is of A-type zero. The converse is also true:

COROLLARY 5.2. *A set P is of* B-*type zero if and only if it is of* A-*type zero.*

Let

$$H(x, t) \cong \sum_0^\infty P_n(x)t^n, \tag{5.12}$$

$$T(x, t) \cong \sum_0^\infty T_n(x)t^n. \tag{5.13}$$

Then (5.11) is seen to be equivalent to the relation

$$tHT = \frac{\partial H}{\partial x},$$

which, when solved for H, gives us

$$H(x, t) = A(t) \exp\left\{t \int_0^x T(x, t)\, dx\right\}, \tag{5.14}$$

where it is to be understood here and later that $A(t)$ is an arbitrary power series beginning with a (non-zero) constant term. Conversely, if T is any series (5.13) where T_n is a polynomial of degree not exceeding n, then $H(x, t)$ as defined by (5.14) is such that (5.11) holds.[30]

If $T(x, t)$ is written as a power series in x rather than t, (5.14) assumes the form

$$H(x, t) \cong A(t) \exp\{xH_1(t) + x^2H_2(t) + \cdots\}, \tag{5.15}$$

[29] This is an extension of Theorem 2.6.

[30] The T_n's are not completely arbitrary. For P to be a set, it is necessary that P_n be of degree exactly n. This reflects itself in the non-vanishing for $n = 1, 2, \cdots$ of certain polynomials in $t_{00}, t_{11}, t_{22}, \cdots$, where t_{ii} is the coefficient of x^i in $T_i(x)$. These conditions can be obtained from (5.11) by demanding that the right member be of degree exactly $(n - 1)$.

where the H_i are power series in t, H_i beginning with a term in t^i or possibly higher. (H_1 definitely begins with a term in t.)

For P to be of B-type k, $T(x, t)$ must be a polynomial in x of degree k. This is necessary and sufficient. On integrating, we get a polynomial of degree $k + 1$ in x. Hence we have

THEOREM 5.3. *A necessary and sufficient condition that a set P be of* B-*type k is that it be given by* (5.12) *where H is of the form*

$$H(x, t) = A(t) \cdot \exp \{xH_1(t) + \cdots + x^{k+1}H_{k+1}(t)\}, \tag{5.16}$$

the $H_i(t)$ being of form[31]

$$H_i(t) \cong h_{ii}t^i + h_{i,i+1}t^{i+1} + \cdots \qquad (h_{11} \neq 0). \tag{5.17}$$

Given a set P, there exists a unique sequence of polynomials $\{U_n(x)\}$, U_n of degree not exceeding n, such that

$$nP_n = U_1P_{n-1} + \cdots + U_nP_0 \qquad (n = 1, 2, \cdots). \tag{5.18}$$

(The U_n's are determined successively if we set $n = 1, 2, \cdots$.)

DEFINITION. *P is of* C-*type k if the maximum degree of the U_n's is* $(k + 1)$.

If we set

$$U(x, t) \cong \sum_0^\infty U_{n+1}(x)t^n, \tag{5.19}$$

then from (5.12) and (5.18),

$$HU = \frac{\partial H}{\partial t},$$

so that

$$H(x, t) = c \cdot \exp\left\{\int_0^t U(x, t)\, dt\right\}. \tag{5.20}$$

Here c is an arbitrary (non-zero) constant. Comparing (5.14) and (5.20), we see that

$$\log c + \int_0^t U(x, t)\, dt = \log A(t) + t\int_0^x T(x, t)\, dx, \tag{5.21}$$

so that

$$\begin{cases} U(x, t) = \dfrac{A'}{A} + \displaystyle\int_0^x \frac{\partial}{\partial t}(tT)\, dx, \\ tT(x, t) = \displaystyle\int_0^t \frac{\partial U}{\partial x}\, dt. \end{cases} \tag{5.22}$$

COROLLARY 5.3. *P is of* C-*type k if and only if it is of* B-*type k.*

[31] The non-vanishing conditions on the t_{ii} referred to in the preceding footnote become non-vanishing conditions on the h_{ii}.

For P is of C-type k if and only if the brace in (5.20) is a polynomial in x of degree $k + 1$. (5.20) thus reduces to (5.16), including the conditions (5.17).

There is no such close link between A-type and B-type. Consider, for example, the set

$$P_n(x) = \frac{x^n}{n!(n+1)!}.$$

It is of A-type *one* since $L[P_n] = P_{n-1}$, where

$$L[y] \equiv 2y' + xy''.$$

(Also, $J_0(t) = 2t$, $J_1(t) = t^2$, and $M_n(t) = (n+1)!t^n$.) On the other hand,

$$H(x, t) = \sum_0^{\infty} \frac{(xt)^n}{n!(n+1)!},$$

so that $\log H$ is decidedly not a polynomial in x. P is therefore of *infinite* B-type.

Let P be any set, and L its associated operator (i.e., $L[P_n] = P_{n-1}$). From (5.18) we have

$$U_1L[P_n] + \cdots + U_nL^n[P_n] = nP_n,$$

so that we get

COROLLARY 5.4. *Every set P satisfies an equation of form*

$$V[y] \equiv \sum_{k=1}^{\infty} U_k L^k[y] = \lambda y, \tag{5.23}$$

where $\lambda = n$ for $y = P_n$. The U_n's are defined as in (5.18), *and L is the operator associated with P. If P is of* B-*type k, the coefficients in* (5.23) *are polynomials of maximum degree* $(k + 1)$.

In connection with sets of finite type (according to one definition or another) there arises the problem of the application of finite type sets to the solution of functional equations. This problem we reserve for another occasion. We shall terminate the present section by showing that the Legendre polynomials are of infinite type according to all the definitions given.

The B-type is determined by the maximum degree of the polynomials $T_n(x)$ of (5.11). Using relations (5.12) to (5.14), where $H = (1 - 2tx + t^2)^{-\frac{1}{2}}$, we find that

$$T(x, t) = \sum T_n(x)t^n = (1 - 2tx + t^2)^{-1}, \tag{a}$$

so that T_n satisfies the recurrence relation

$$T_n - 2xT_{n-1} + T_{n-2} = 0, \qquad n > 0. \tag{b}$$

With the initial conditions $T_0 = 1$, $T_1 = 2x$, the solution of (b) is

$$T_n(x) = \frac{1}{2\theta}\{(x+\theta)^{n+1} - (x-\theta)^{n+1}\}, \qquad \theta = (x^2 - 1)^{\frac{1}{2}}. \tag{c}$$

It is seen that $T_n(x)$ is of degree n, so that $\{X_n(x)\}$ is of infinite B-type (and C-type).

Now consider the A-type of $\{X_n\}$. If L is the associated operator then,

$$\text{(d)} \qquad L[X_n] = X_{n-1},$$

where

$$\text{(e)} \qquad L[y] \equiv L_0(x)y' + L_1(x)y'' + \cdots .$$

Multiplying (d) by t^n and summing from $n = 0$ to $n = \infty$, we obtain

$$\text{(f)} \qquad L[H] = tH, \qquad H = (1 - 2tx + t^2)^{-\frac{1}{2}}.$$

Relation (e), for $y = H$, simplifies to

$$\text{(g)} \qquad t = \sum_0^\infty L_n(x) \frac{1 \cdot 3 \cdots (2n+1)t^{n+1}}{(1 - 2tx + t^2)^{n+1}};$$

and if we set

$$\text{(h)} \qquad \lambda = \frac{t}{1 - 2tx + t^2},$$

this becomes

$$\text{(i)} \quad \frac{1}{2\lambda}\{1 + 2x\lambda - (1 + 4x\lambda + 4(x^2 - 1)\lambda^2)^{\frac{1}{2}}\} = \sum_0^\infty 1 \cdot 3 \cdots (2n+1)\lambda^{n+1} L_n(x).$$

Since (g) is an identity in the variables t, x, so is (i) an identity in the variables λ, x. If $\{X_n\}$ is of finite A-type, the right member, and therefore the left, is a polynomial in x. This is manifestly untrue. Hence $\{X_n\}$ is of infinite A-type.

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